Linear and Exponential Functions

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Linear and Exponential functions are the two most common functions that professionals will encounter in business, accounting, and finance courses. Even in most fields of science, these are common, if not the most common, function. So it is reasonable to ask what are they and what do they mean?

Linear functions are defined by: $y = f(t) = m\,t + b$

Exponential functions are defined by: $y = g(t) = C\,a^t$

Where t is the independent variable, and y is the dependent variable. For simplicity we will consider the independent variable, t, to be time. This is not a requirement, but is a common use of this type of function in business, accounting, and finance.

So under what circumstances would someone use a <u>linear function</u>? When an amount, represented by the dependent variable increases by a certain amount over every time unit. So for example, if you are getting paid by the hour and receiving a wage, then you will be getting paid a certain amount of money for every hour you worked. If you were making $10/hour the equation that would indicate your salary would be:

$y = 10\,t$,
where y = amount paid, and t = number of hours worked. The "m" in this case is $10/hour and b = 0.

Other applications:

- Flow of water coming steadily from a hose. In this case the "m" would have units like gallons/hour.
- Amount of painting (of a wall). In this case the "m" would have units like square feet/hour.
- Amount of distance traveled (when traveling at a steady rate). In this case "m" would have units like miles/hour.
- If you are handing out tickets at a steady rate, then the number left would be a linear equation. In this case "m" would have units like number/hour and "b" would be the number of tickets you had before handing any out.

The key characteristic of the linear equation is that it represents something that is either increasing or decreasing by a certain amount every time period. The <u>ratio</u> of the amount from one unit time to the next will not be the same.

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So under what circumstances would you use an exponential equation? An exponential equation models a process that changes by a multiplicative factor (or a fixed ratio) over a time unit. For example, a compound investment increases by a factor every period. If the period is one year (the common period for investments), then the base is 1 + the decimal equivalent of the percentage or:

$a = 1 + r$

$C(t) = C_o (1 + r)^t$ where C_o is the initial investment, r is the decimal equivalent to the R percentage.

The factor "a" can be expressed indirectly in several ways:

- A percent (or fraction) increase might be specified, then $a = 1+ r$ where r is the decimal equivalent of the percentage (or fraction). The increase could be phrased as a gain. For example, a 15% gain from one time period to the next would be expressed as $a = 1 + 0.15 = 1.15$
- A percent (or fraction) decrease might be specified, then $a = 1- r$ where r is the decimal equivalent of the percentage (or fraction). This decrease could be phrased as a "loss." For example, a 15% loss from one time period to the next would be expressed as $a = 1 -0.15 = 0.85$
- The multiplicative amount, f, over an arbitrary time – T could be specified. Then $a= f^{1/T}$ so the resulting exponential function would be $y = C (f^{t/T})$.
  - One common example is half-life such as radioactive decay. In this case the $f = ½$ and the $T = t_{halflife}$
  - Another common example is doubling times such as bacterial growth. In this case the $f = 2$ and the $T = t_{doubliingtime}$
- For compounding investments over definite intervals (n), $a = (1+r/n)^n$ so the equation becomes $y = C (1+r/n)^{nt}$. If the compounding becomes continuous, then the equation changes to $y = Ce^{rt}$.
- The "r" in the above equation can be interpreted as the instantaneous rate of change per unit amount.

The key characteristic of an exponential equation is that it involves an increase or decrease that is specified by a multiplicative factor or ratio. The amount of change from one unit of time to the next will not be constant.